LOCAL SMOOTHING FOR THE BACKSCATTERING TRANSFORM

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ABSTRACT. An analysis of the backscattering data for the Schrödinger operator in odd dimensions $n \geq 3$ motivates the introduction of the backscattering transform $B : C_0^\infty(\mathbb{R}^n;\mathbb{C}) \to C^\infty(\mathbb{R}^n;\mathbb{C})$. This is an entire analytic mapping and we write $Bv = \sum_1^\infty B_N v$ where $B_N v$ is the N:th order term in the power series expansion at $v = 0$. In this paper we study estimates for $B_N v$ in $H_{(s)}$ spaces, and prove that Bv is entire analytic in $v \in H_{(s)} \cap \mathcal{E}'$ when $s \geq (n-3)/2$.

1. INTRODUCTION

The present note is devoted to proving continuity and smoothing properties of the backscattering transform for the Schrödinger operator in odd dimensions $n > 1$.

In order to state the main result a brief description of the mathematical objects involved is necessary. (The reader is referred to [10], [9], [8] for details.)

Consider the Schrödinger operator $H_v = -\Delta + v$ in $\mathbb{R}^n$, where $v \in L^2_{\text{cpt}}(\mathbb{R}^n)$. Assume that H_v with domain $H_{(2)}(\mathbb{R}^n)$ is self-adjoint and the wave operators

$$W_\pm = \lim_{t\to\pm\infty} e^{\mathrm{i}tH_v} e^{-\mathrm{i}tH_0}$$

exist. Then the operator vW_+ is continuous from L^2 to L^1, and therefore its distribution kernel $v(x)W_+(x,y)$ is defined. After composing it with a non-singular linear transformation, we arrive at the distribution $v(x-y)W_+(x-y,x+y)$ in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$. Since v is compactly supported we may integrate with respect to y and obtain the distribution

$$2^n \int v(x-y)W_+(x-y,x+y)\,\mathrm{d}y.$$

(The normalization factor here is introduced in order that the expression above should be equal to $v(x)$ when W_+ is replaced by the identity.) It was proved in [10] that when $v \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$ the integral above represents the inverse Fourier transform of the backscattering part of the scattering matrix, when this is represented as a function in the momentum variables. The real part of the expression above is equal to

$$\beta v(x) = 2^n \int v(x-y)W(x-y,x+y)\,\mathrm{d}y,$$

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where the operator $W = (W_+ + W_-)/2$ has a real-valued distribution kernel.

The backscattering transform Bv of $v \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$ is a slight modification of βv. Let $K_v(t)$ be the wave group associated to the operator

$$\Box_v = \partial_t^2 - \Delta_x + v,$$

i.e., $u(x,t) = (K_v(t)f)(x)$ is, for every $f \in C_0^\infty(\mathbb{R}^n)$, the unique solution in $C^1([0,\infty), L^2(\mathbb{R}^n))$ to the Cauchy problem

$$\Box_v u(x,t) = 0, \quad u(x,0) = 0, \quad (\partial_t u)(x,0) = f(x).$$

Then $K_v(t)$ is a strongly continuous function of t with values in the space of bounded linear operators on $L^2(\mathbb{R}^n)$. (See [9] for details.) We have that $|x-y| \le t$ in the support of $K_v(x,y;t)$ and $|x-y| = t$ in the support of $K_0(x,y;t)$. This ensures that the operator

$$G = -\int_0^\infty K_v(t) v \dot{K}_0(t)\,\mathrm{d}t$$

is well-defined and continuous on $L^2_{\mathrm{cpt}}(\mathbb{R}^n)$, where the dot denotes derivative in the variable t. Theorem 7.1 in [10] gives the relation between G and W above: There exist an orthonormal basis $(f_j)_{1\le j\le\mu}$ of real eigenfunctions corresponding to the negative part of the spectrum of H_v and a set $(g_j)_{1\le j\le\mu}$ of smooth real-valued functions such that

$$W = I + G + \sum_1^\mu f_j \otimes g_j.$$

It turns out (see below) that $G = G_v$, considered as function of v with values in the space of continuous linear operators in $L^2_{\mathrm{cpt}}(\mathbb{R}^n)$, extends to an entire analytic function of $v \in C_0^\infty(\mathbb{R}^n)$, i.e., to the space of complex-valued v in C_0^∞. Also, if v is sufficiently small (in a sense that we do not make precise here), there are no bound states and $W = I + G$ then. For these reasons it is natural to modify the definition of βv by subtracting the contribution from $\sum_1^\mu f_j \otimes g_j$.

Definition. Assume $v \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. The backscattering transform Bv of v is defined by

$$(Bv)(x) = v(x) + 2^n \int v(x-y) G(x-y, x+y)\,\mathrm{d}y.$$

Here the integral is taken in distribution sense and $v(x)G(x,y)$ is the distribution kernel of the operator vG.

It was proved in [9] that $G = G_v$ extends to an entire analytic function of $v \in L^q_{\mathrm{cpt}}(\mathbb{R}^n)$ when $q > n$. For such v we can define Bv again as in the previous definition and Bv will be entire analytic in v with values in $\mathcal{D}'(\mathbb{R}^n)$. We write

$$Bv = \sum_1^\infty B_N v$$

where $B_N v$ is the N:th order term in the power series expansion at $v = 0$. There are other spaces of v (containing C_0^∞ as a dense subset) to which Bv can be extended analytically. For reasons of continuity such expansions can be studied by deriving estimates for the $B_N v$ when $v \in C_0^\infty$. In this paper we shall study estimates for $B_N v$ in $H_{(s)}$ spaces, and prove that Bv is entire analytic in $v \in H_{(s)} \cap \mathcal{E}'$ when $s \geq (n-3)/2$.

We recall some basic ingredients in the construction of Bv when $v \in C_0^\infty(\mathbb{R}^n)$. We recall from [9], or section 11 in [10], that

$$K_v(t) = \sum_{N \geq 0} (-1)^N K_N(t), \tag{1.1}$$

where K_N are inductively defined by

$$\begin{aligned} K_0(t) &= \frac{\sin t|D|}{|D|}, \\ K_N(t) &= (K_{N-1} * vK_0)(t) = \int_0^t K_{N-1}(s) v K_0(t-s)\,\mathrm{d}s, \quad N \geq 1. \end{aligned} \tag{1.2}$$

One has the estimate

$$\|K_N(t)\|_{L^2 \to L^2} \leq \|v\|_{L^\infty}^N t^{2N+1}/(2N+1)!$$

Since the distribution kernel $K_N(x, y; t)$ of $K_N(t)$ is supported in the set where $|x - y| \leq t$, it makes sense to consider

$$G_N = (-1)^N \int_0^\infty K_{N-1}(t) v \dot{K}_0(t)\,\mathrm{d}t. \tag{1.3}$$

This is a continuous linear operator in $L^2_{\mathrm{cpt}}(\mathbb{R}^n)$, and the estimates for the K_N show that

$$G = \sum_1^\infty G_N$$

is an entire analytic function of v. We see that

$$(B_N v)(x) = 2^n \int v(y) G_{N-1}(y, 2x - y)\,\mathrm{d}y, \quad N \geq 2. \tag{1.4}$$

The following theorem (Theorem 8, [9]) reveals the smoothing properties of B_N for large N.

Theorem 1.1. *Let $q > n$ and k be a nonnegative integer. Then there is a positive integer $N_0 = N_0(n, q, k)$ such that $\Delta^k B_N v \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ when $v \in L^q(\mathbb{R}^n)$ has compact support and $N \geq N_0$. Moreover, if $R_1, R_2 > 0$, there is a constant C, depending on n, k, R_1, R_2 and q only such that*

$$\|\Delta^k B_N v\|_{L^2(B(0,R_1))} \leq C^N \|v\|_{L^q}^N / N!, \qquad N \geq N_0,$$

whenever $v \in L^q(\mathbb{R}^n)$ has support in the ball $B(0, R_2)$.

The aim of this paper is to study (local) continuity properties of the operators B_N in $H_{(s)}$ spaces.

Let $\|\cdot\|_{(s)}$ denote the norm on the Sobolev space $H_{(s)}(\mathbb{R}^n)$. Also $H_{(s)}(\Omega)$, $s \geq 0$, is the space of functions which are restrictions to Ω of functions from the Sobolev space $H_{(s)}(\mathbb{R}^n)$, when Ω is an open set with smooth boundary. The norm on $H_{(s)}(\Omega)$, $s \geq 0$, is the quotient norm

$$\|f\|_{H_{(s)}(\Omega)} = \inf\{\|F\|_{(s)}; F \in H_{(s)}(\mathbb{R}^n), F = f \,\mathrm{in}\, \Omega\}.$$

Our main result here is contained in the next theorem.

Theorem 1.2. *Assume $0 \leq a \leq s - (n-3)/2$, and let $N(a,s)$ be the smallest integer N such that $a < N-1$ and $a \leq (N-1)(s-(n-3)/2)$. Then there is a constant C, which depends on n, s and a only, such that*

$$\|B_N v\|_{H_{(s+a)}(B(0,R))} \leq C^N R^{(N-1)/2} N^{-N/2} \|v\|_{(s)}^N$$

when $N \geq N(a,s)$, $R > 0$ and $v \in C_0^\infty(B(0,R))$.

A first corollary of this result is the above-mentioned analyticity of the backscattering transformation.

Corollary 1.3. *The mapping $C_0^\infty(\mathbb{R}^n) \ni v \to Bv \in C^\infty(\mathbb{R}^n)$ extends to an entire analytic mapping from $H_{(s)}(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$ to $H_{(s),\mathrm{loc}}(\mathbb{R}^n)$ whenever $s \geq (n-3)/2$.*

A second corollary gives the regularity of the difference between v and its backscattering transform.

Corollary 1.4. *Assume $s \geq (n-3)/2$ and $0 \leq a < 1$ satisfy $a \leq s-(n-3)/2$. If $v \in H_{(s)}(\mathbb{R}^n)$ is compactly supported, then*

$$v - Bv \in H_{(s+a),\mathrm{loc}}(\mathbb{R}^n). \tag{1.5}$$

The outline of this note is as follows. In the next section we derive a formula that generalizes to arbitrary $N > 2$ the formula

$$(B_2 v)(x) = \int\limits_{(\mathbb{R}^n)^2} E_2(y_1, y_2) v(x - \frac{y_2 - y_1}{2}) v(x - \frac{y_1 + y_2}{2}) \,\mathrm{d}y_1 \,\mathrm{d}y_2,$$

which appears in Corollary 10.7 of [10]. Here E_2 is the unique fundamental solution of the ultra-hyperbolic operator $\Delta_x - \Delta_y$ such that $E_2(x,y) = -E_2(y,x)$ and E_2 is rotation invariant separately in x and y. When $N > 2$ we have to replace E_2 by a distribution $E_N \in \mathcal{D}'((\mathbb{R}^n)^N)$ which is a fundamental solution of the operator $P_N = (\Delta_{x_N} - \Delta_{x_1})(\Delta_{x_N} - \Delta_{x_2}) \cdots (\Delta_{x_N} - \Delta_{x_{N-1}})$. The distribution E_N is discussed in more detail in Section 3.

Once these formulas have been obtained, the proof of the theorem becomes elementary. The third section contains estimates of the Fourier transforms of (cut-offs of) E_N. These are in turn used in the fourth section when the estimates in Theorem 1.2 are obtained by Fourier transforming the formula for $B_N v$.

We close this presentation with a few words on the existing literature on backscattering problems for the potential scattering in odd dimensions. The backscattering map was studied also in [1] for dimension 3 and in [3] for arbitrary dimensions, and local uniqueness was proved for potentials in a certain weighted Hölder space. The actual backscattering transform defined as above was considered in [7] for dimension 3, and it was proved to be analytic when defined on small potentials v such that $\nabla v \in L^1$ and with values in the same space, and consequently uniqueness for the inverse backscattering problem was obtained for small potentials in this space. Generic uniqueness was proved in [13] for compactly supported bounded potentials in dimension 3. We also mention [14] for an approach using Lax-Phillips scattering. The problem of recovering the singularities of v from the backscattering data was considered in [4], [6] and [12]. Our result here improves the results in [12] in the sense that it shows that the difference between the potential v and its backscattering transform is more regular and the result holds for arbitrary odd $n \geq 3$.

Finally, let us fix some notation we use throughout the paper. If $N \geq 2$ we use the notation $\vec{x} = (x_1, \dots, x_N) \in (\mathbb{R}^m)^N$ where $x_1, \dots x_N \in \mathbb{R}^m$, for m a positive integer. If $x \in \mathbb{R}^m$ we shall set $\langle x \rangle = (1 + |x|^2)^{1/2}$. The Fourier transform of a distribution u will be denoted either by $\hat{u}$ or by $\mathcal{F}u$.

2. A formula for B_N

In this section we are going to write $B_N v$ as the value at $(v, \dots, v)$ of a N-linear operator defined from $C_0^\infty(\mathbb{R}^n) \times \cdots \times C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$, following the procedure in [9]. The key point here is the fact that $K_0(t)$ obeys Huygens' principle, more specifically, that its convolution kernel $k_0(x; t)$ is supported in the set where $|x| = t$.

When $N = 1, 2, \dots$ we define $Q_N \in \mathcal{D}'((\mathbb{R}^n)^N \times \mathbb{R}_+)$ inductively by

$$Q_1(x; t) = k_0(x; t), \tag{2.1}$$

$$Q_N(x_1, \dots, x_N; t) = \int_0^t Q_{N-1}(x_1, \dots x_{N-1}; t - s) Q_1(x_N; s) \, \mathrm{d}s \quad \text{when } N \geq 2. \tag{2.2}$$

Then the mapping

$$\mathbb{R}_+ \ni t \to Q_N(x_1, \dots x_N; t) \in \mathcal{D}'((\mathbb{R}^n)^N)$$

is smooth when $N \geq 1$. It is easily seen that Q_N is symmetric in $x_1, \dots x_N$, rotation invariant separately in these variables and, since $|x| = t$ in the support of $k_0(x,t)$, it follows that

$$|x_1| + \dots + |x_N| = t \quad \text{in} \quad \operatorname{supp} Q_N. \tag{2.3}$$

Next we define $E_N \in \mathcal{D}'((\mathbb{R}^n)^N)$, $N \geq 2$, by

$$E_N(x_1, \dots, x_N) = (-1)^{N-1} \int_0^\infty Q_{N-1}(x_1, \dots, x_{N-1}; t) \dot{k}_0(x_N; t) \, \mathrm{d}t. \tag{2.4}$$

It follows from (2.3) that

$$|x_1| + \dots + |x_{N-1}| = |x_N| \quad \text{in} \quad \operatorname{supp} E_N, \tag{2.5}$$

E_N is rotation invariant separately in all variables, and symmetric in $x_1, \dots, x_{N-1}$. We recall here that

$$E_2(x,y) = 4^{-1}(\mathrm{i}\pi)^{1-n} \delta^{(n-2)}(x^2 - y^2) \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R}^n$$

is the unique fundamental solution of the ultra-hyperbolic operator $\Delta_x - \Delta_y$ such that $E_2(x,y) = -E_2(y,x)$ and E_2 is rotation invariant separately in x and y. (See Theorem 10.4 and Corollary 10.2 in [10].)

The next lemma follows easily from (1.2) and (1.3) by induction and some simple computations.

Lemma 2.1. *Assume $v \in C_0^\infty(\mathbb{R}^n)$. Then*

$$K_N(x,y;t) = \int v(x_1) \cdots v(x_N) Q_{N+1}(x - x_1, x_1 - x_2, \dots, x_{N-1} - x_N, x_N - y; t) \, \mathrm{d}\vec{x}, \tag{2.6}$$

$$G_N(x,y) = \int_{(\mathbb{R}^n)^N} v(x_1) \cdots v(x_N) E_{N+1}(x - x_1, x_1 - x_2, \dots, x_{N-1} - x_N, x_N - y) \, \mathrm{d}\vec{x} \tag{2.7}$$

for every $N \geq 1$.

Proposition 2.2. *For $N \geq 2$*

$$(B_N v)(x) = \int_{(\mathbb{R}^n)^N} E_N(y_1, \dots, y_N) v(x - \frac{y_N}{2} - Y_0) v(x - \frac{y_N}{2} - Y_1) \cdots v(x - \frac{y_N}{2} - Y_{N-1}) \, \mathrm{d}\vec{y}$$

when $v \in C_0^\infty(\mathbb{R}^n)$, where

$$Y_0 = \frac{1}{2} \sum_{j=1}^{N-1} y_j \quad \textit{and} \quad Y_k = Y_0 - \sum_{j=1}^{k} y_j, \quad 1 \leq k \leq N-1.$$

Proof. We use (2.7) to express $G_{N-1}(y, 2x - y)$ in (1.4) and get thus

$$(B_N v)(x) = 2^n \int_{\mathbb{R}^n \times (\mathbb{R}^n)^{N-1}} v(y) v(x_1) \cdots v(x_{N-1})$$
$$E_N(y - x_1, x_1 - x_2, \dots, x_{N-2} - x_{N-1}, x_{N-1} + y - 2x) \, \mathrm{d}y \, \mathrm{d}\vec{x}.$$

The proposition follows by changing variables $y-x_1=-y_1$, $x_1-x_2=-y_2$, $\dots$, $x_{N-2}-x_{N-1}=-y_{N-1}$, $x_{N-1}+y-2x=-y_N$, hence

$$\begin{gathered}
y = x - \frac{1}{2}\sum_{j=1}^{N} y_j = x - \frac{y_N}{2} - Y_0 \\
x_1 = y + y_1 = x - \frac{y_N}{2} - Y_1 \\
\dots \\
x_{N-1} = x_{N-2} + y_{N-1} = x - \frac{y_N}{2} - Y_{N-1}.
\end{gathered}$$

Here we have made use of the invariance properties of E_N, which in particular ensure that $E_N(y_1,\dots,y_N)$ is even in each y_j. □

3. The distribution E_N

We need some further information on the distribution E_N defined in (2.4).

The first result is a characterization of E_N. We denote

$$P_N = (\Delta_1 - \Delta_N)\cdots(\Delta_{N-1} - \Delta_N),$$

where Δ_j in the Laplacian in the variables x_j.

Lemma 3.1. *The distribution E_N is a fundamental solution of P_N. It has the following properties:*

(i) $E_N(x_1,\dots,x_N)$ *is rotation invariant in each* x_j*;*
(ii) $|x_1|+\cdots+|x_{N-1}|=|x_N|$ *in the support of* E_N*;*
(iii) E_N *is homogeneous of degree* $2(N-1)-nN$.

If E is a fundamental solution of P_N that satisfies (i)-(iii)*, then $E=E_N$.*

Proof. We first prove that $P_NE_N=\delta(x_1,\dots,x_N)$, and when doing this we may assume that $N\ge 3$. Since $\partial_t^2 k_0(x;t)=\Delta_x k_0(x;t)$, it follows easily from (2.2) with N replaced by $N-1$ that

$$\begin{gathered}
\partial_t^2 Q_{N-1}(x_1,\dots,x_{N-1};t) = \Delta_{N-1}Q_{N-1}(x_1,\dots,x_{N-1};t) \\
+Q_{N-2}(x_1,\dots,x_{N-2};t)\delta(x_{N-1}).
\end{gathered}$$

It follows from (2.4) then that

$$\Delta_N E_N(x_1,\dots,x_N) = (-1)^{N-1}\int_0^\infty Q_{N-1}(x_1,\dots,x_{N-1};t)\partial_t^2\dot{k}_0(x_N;t)\,\mathrm{d}t$$

$$= (-1)^{N-1}\int_0^\infty (\partial_t^2 Q_{N-1}(x_1,\dots,x_{N-1};t))\dot{k}_0(x_N;t)\,\mathrm{d}t$$

$$= (-1)^{N-1}\Delta_{N-1}\int_0^\infty Q_{N-1}(x_1,\dots,x_{N-1};t)\dot{k}_0(x_N;t)\,\mathrm{d}t$$

$$+(-1)^{N-1}\int_0^\infty Q_{N-2}(x_1,\dots,x_{N-2};t)\delta(x_{N-1})\dot{k}_0(x_N;t)\,\mathrm{d}t$$

$$= \Delta_{N-1}E_N(x_1,\dots,x_N) - E_{N-1}(x_1,\dots,x_{N-2},x_N)\delta(x_{N-1}).$$

We have proved therefore that

$$(\Delta_{N-1}-\Delta_N)E_N(x_1,\dots,x_N) = E_{N-1}(x_1,\dots,x_{N-2},x_N)\delta(x_{N-1}). \tag{3.1}$$

Assuming, as we may, that the assertion has been proved for lower values of N and letting $(\Delta_1 - \Delta_N)\cdots(\Delta_{N-2}-\Delta_N)$ act on both sides of (3.1) we may conclude that $P_N E_N(x_1,\dots,x_N) = \delta(x_1,\dots x_N)$.

The conditions (i) and (ii) are simple consequences of the definitions, together with the fact that $k_0(x;t)$ is rotation invariant in x and supported in the set where $|x| = t$. Since k_0 is homogeneous when considered as a distribution in x and t, it follows that E_N is a homogeneous distribution. Its degree of homogeneity must be equal to the degree of P_N minus the dimension of $(\mathbb{R}^n)^N$. This proves (iii).

It remains to prove that $\Phi = 0$ if $\Phi = \Phi(x_1,\dots,x_N)$ is a distribution satisfying the conditions in (i) -(iii) and $P_N\Phi = 0$.

Define

$$\Psi(x_1,\dots,x_N) = (\Delta_1-\Delta_N)\cdots(\Delta_{N-2}-\Delta_N)\Phi(x_1,\dots,x_N)$$

(with the interpretation $\Psi = E_2$ if $N = 2$). This a homogeneous distribution of degree $2 - nN$ and

$$(\Delta_{N-1}-\Delta_N)\Psi = 0.$$

Since Ψ is rotation invariant in each x_j, it follows from Theorem 10.1 of [10] that Ψ is symmetric in x_{N-1}, x_N. Since $|x_1| + \cdots + |x_{N-1}| = |x_N|$ in the support of Ψ this implies that $x_1 = \cdots = x_{N-2} = 0$ in its support. Hence

$$\Psi(x_1,\dots,x_N) = \sum \delta^{(\alpha)}(x_1,\dots x_{N-2})u_\alpha(x_{N-1},x_N),$$

where the $u_\alpha(x,y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ are solutions to the ultra-hyperbolic equation. The rotation invariance of Φ in the x_j implies that the summation takes place over even $|\alpha|$ only and that the $u_\alpha(x,y)$ and rotation invariant separately in x and y. Also, $u_\alpha(x,y) = u_\alpha(y,x)$ and u_α is homogeneous of degree μ_α, where

$$\mu_\alpha = 2 - nN + (N-2)n + |\alpha| = 2 + |\alpha| - 2n$$

is even. Since $\mu_\alpha > -2n$ the proof is completed if we prove that u_α vanishes outside the origin in $\mathbb{R}^n \times \mathbb{R}^n$. In this set we may view u_α as a function $f(s,t)$ in $s = |x|, t = |y|$. Since it is supported in the set where $s = t$ we may write

$$f(s,t) = \sum_{0 \le j \le J} c_j \delta^{(j)}(s-t)(s+t)^{j+\nu}$$

where $\nu = 1 + \mu_\alpha$ is odd, and the summation takes place over even j only, since $f(s,t) = f(t,s)$. We assume that $f \neq 0$ and shall see that this leads to a contradiction.

Assume now that $c_J \neq 0$. Expressing the Laplacian in polar coordinates, we get the equation

$$0 = \Big(\partial_s^2 - \partial_t^2 + (n-1)(s^{-1}\partial_s - t^{-1}\partial_t)\Big) f(s,t).$$

The right-hand side here is a linear combination of $\delta^{(j)}(s-t)(s+t)^{j+\nu-2}$ with $j \le J+1$, and a simple computation shows that the coefficient in front of $\delta^{(J+1)}(s+t)^{J+\nu-1}$ is equal to $4c_J\kappa$, where

$$\kappa = (J+\nu) + n - 1.$$

This gives us a contradiction, since we know that $\kappa = 0$ while the right-hand side above is an odd integer. We have proved therefore that u_α vanishes outside the origin. $\square$

We need to establish estimates for the Fourier transforms of certain cut-offs of E_N. Namely, we shall consider distributions of the form

$$(-1)^{N-1} \int_0^\infty Q_{N-1}(x_1, \dots, x_{N-1}; t) \dot{k}_0(x_N; t) \chi(t)\, \mathrm{d}t, \qquad N = 2, 3, \dots, \tag{3.2}$$

where $\chi \in C_0^\infty(\mathbb{R})$. We notice that $|x_1| + \cdots + |x_{N-1}| = |x_N| < R_0$ in the support of this distribution whenever the support of χ is contained in the interval $(-\infty, R_0)$. Also, if $\chi(t) = 1$ when $0 \le t \le R_1$, then the restrictions to $(\mathbb{R}^n)^{N-1} \times B(0, R_1)$ of the distribution in (3.2) and of E_N coincide.

We start with some preparatory computations. When $a \in \mathbb{R}$ define

$$\varphi_a(t) = Y_+(t) \frac{\sin(ta)}{a}, \qquad t \in \mathbb{R},$$

where Y_+ is the Heaviside's function.

Lemma 3.2. *Assume $N \geq 2$ and $a_1, \dots a_N$ are real numbers such that $a_j^2 \neq a_k^2$ when $j \neq k$. Then we have the identity*

$$(\varphi_{a_1} * \cdots * \varphi_{a_N})(t) = \sum_{j=1}^{N} \prod_{k \neq j} \frac{1}{a_k^2 - a_j^2} \varphi_{a_j}(t). \tag{3.3}$$

Proof. Let $\varepsilon > 0$ and define $\psi_j(t) = e^{-\varepsilon t} \varphi_{a_j}(t)$. A simple computation shows that

$$\widehat{\psi}_j(\tau) = \frac{1}{(\varepsilon + \mathrm{i}\tau)^2 + a_j^2}.$$

If $\Psi = \psi_1 * \cdots * \psi_N$ it follows that

$$\begin{aligned} \widehat{\Psi}(\tau) = \prod_{1}^{N} \frac{1}{(\varepsilon + \mathrm{i}\tau)^2 + a_j^2} &= \sum_{j=1}^{N} \Big(\prod_{k \neq j} \frac{1}{a_k^2 - a_j^2} \Big) \frac{1}{(\varepsilon + i\tau)^2 + a_j^2} \\ &= \sum_{j=1}^{N} \Big(\prod_{k \neq j} \frac{1}{a_k^2 - a_j^2} \Big) \widehat{\psi}_j(\tau) \end{aligned}$$

Hence

$$\Psi(t) = \sum_{j=1}^{N} \Big(\prod_{k \neq j} \frac{1}{a_k^2 - a_j^2} \Big) \psi_j(t).$$

The lemma then follows when ε tends to 0. □

Lemma 3.3. *When $N \geq 2$, $a_1, \dots a_N \in \mathbb{R}$, $\sigma \in \mathbb{C}$, $\operatorname{Re} \sigma > 0$, define*

$$F(a_1, \dots, a_N; \sigma) = \int_0^\infty (\varphi_{a_1} * \cdots * \varphi_{a_{N-1}})(t) \cos(t a_N) \mathrm{e}^{-\sigma t} \, \mathrm{d}t.$$

Then

$$F(a_1, \dots, a_N; \sigma) = \frac{1}{2} \left(\prod_{1 \leq j \leq N-1} \frac{1}{a_j^2 - (a_N - \mathrm{i}\sigma)^2} + \prod_{1 \leq j \leq N-1} \frac{1}{a_j^2 - (a_N + \mathrm{i}\sigma)^2} \right). \tag{3.4}$$

Proof. Since both sides of (3.4) depend continuously in $a_1, \dots, a_N \in \mathbb{R}$ it is no restriction to assume that $a_j^2 \neq a_k^2$ when $j \neq k$.

First notice that when a, $b \in \mathbb{R}$ and $\sigma \in \mathbb{C}$, $\operatorname{Re} \sigma > 0$, one has

$$\int_0^\infty \varphi_a(t) \cos(tb) \mathrm{e}^{-\sigma t} \, \mathrm{d}t = \frac{a^2 - b^2 + \sigma^2}{(a^2 - b^2 + \sigma^2)^2 + 4b^2\sigma^2}. \tag{3.5}$$

When $N = 2$ (3.4) follows directly from this formula.

Assume $N \geq 3$. The previous lemma and (3.5) give

$$F(a_1, \dots, a_N; \sigma) = \sum_{j=1}^{N-1} \prod_{k \neq j} \frac{1}{a_k^2 - a_j^2} \int_0^\infty \varphi_{a_j}(t) \cos(t a_N) \mathrm{e}^{-\sigma t} \, \mathrm{d}t$$
$$= \sum_{j=1}^{N_1} \Big(\prod_{k \neq j} \frac{1}{a_k^2 - a_j^2} \Big) \frac{a_j^2 - a_N^2 + \sigma^2}{(a_j^2 - a_N^2 + \sigma^2)^2 + 4 a_N^2 \sigma^2}.$$

We can simplify this expression by writing

$$t_j = a_j^2 - a_N^2 + \sigma^2, \quad 0 \leq j \leq N-1, \qquad \text{and} \quad b = 2 a_N \sigma.$$

Then

$$F(a_1, \dots, a_N; \sigma) = \sum_{j=1}^{N-1} \Big(\prod_{k \neq j} \frac{1}{t_k - t_j} \Big) \frac{t_j}{t_j^2 + b^2}$$
$$= \frac{1}{2} \sum_{j=1}^{N-1} \Big(\prod_{k \neq j} \frac{1}{t_k - t_j} \Big) \frac{1}{t_j - \mathrm{i}b} + \frac{1}{2} \sum_{j=1}^{N} \Big(\prod_{k \neq j, k \leq N} \frac{1}{t_k - t_j} \Big) \frac{1}{t_j + \mathrm{i}b}$$
$$= \frac{1}{2} \prod_{1 \leq j \leq N} \frac{1}{t_j - \mathrm{i}b} + \frac{1}{2} \prod_{1 \leq j \leq N} \frac{1}{t_j + \mathrm{i}b}.$$

This finishes the proof of the lemma, after noticing that $t_j \pm \mathrm{i}b = a_j^2 - (a_N \mp \mathrm{i}\sigma)^2$. $\square$

The next lemma is a direct consequence of Theorem 1.4.2 in [5].

Lemma 3.4. *There is a sequence $(\chi_N)_1^\infty$ in $C_0^\infty(\mathbb{R})$ such that $\chi_N(t) = 1$ when $|t| \leq 1$, $\chi_N(t) = 0$ when $|t| > 2$ and*

$$|\chi_N^{(k)}(t)| \leq C^k N^k, \quad 0 \leq k \leq 2N+2.$$

Here $C > 0$ is independent of N.

In what follows R is an arbitrary positive number. We set $\chi_{N,R}(t) = \chi_N(t/R)$, so that $\chi_{N,1} = \chi_N$. We define

$$E_{N,R} = (-1)^{N-1} \int_0^\infty Q_{N-1}(x_1, \dots, x_{N-1}; t) \dot{k}_0(x_N; t) \chi_{N,R}(t) \, \mathrm{d}t, \quad N = 2, 3 \dots. \tag{3.6}$$

We notice that

$$|x_1| + \dots + |x_{N-1}| = |x_N| \leq 2R \quad \text{in } \operatorname{supp}(E_{N,R}) \tag{3.7}$$

and

$$E_{N,R}(x_1, \dots, x_N) = E_N(x_1, \dots, x_N) \quad \text{when } |x_N| \leq R. \tag{3.8}$$

We shall derive estimates for the Fourier transform $\mathcal{F}E_{N,R}(\xi_1,\dots,\xi_N)$ of $E_{N,R}$. We notice here that, due to the homogeneity of $Q_{N-1}(\cdot;t)$ and of $\dot{k}_0(\cdot;t)$ and to the definition of $\chi_{N,R}$, we have

$$E_{N,R}(Rx_1,\dots,Rx_N) = R^{2N-2}R^{-Nn}E_{N,1}(x_1,\dots x_N).$$

It follows that

$$(\mathcal{F}E_{N,R})(\xi_1,\dots,\xi_N) = R^{2N-2}\mathcal{F}E_{N,1}(R\xi_1,\dots,R\xi_N). \tag{3.9}$$

Therefore it is enough to establish estimates for $\mathcal{F}E_{N,1}$.

The distribution $E_{N,1}(x_1,\dots,x_N)$ is rotation invariant in the variables $x_1,\dots,x_N$ and compactly supported. The Fourier transform $\mathcal{F}E_{N,1}(\xi_1,\dots,\xi_N)$ of $E_{N,1}$ is smooth and rotation invariant in each variable ξ_j. We define $F_N(r_1,\dots,r_N)$ when $r_j\ge 0$ by

$$(\mathcal{F}E_{N,1})(\xi_1,\dots,\xi_N) = F_N(r_1,\dots,r_N) \quad \text{when } r_j=|\xi_j|. \tag{3.10}$$

Hence we need estimates of F_N.

Consider $\gamma>0$. Let us define the functions $h_\gamma(r,s)$ through

$$h_\gamma(r,s) = (\gamma+|r-s|)^{-1}(\gamma+|r+s|)^{-1}.$$

Lemma 3.5. *When $s,\ t\in\mathbb{R}$, one has*

$$1+|s-t| \ge \frac{1+|s|}{1+|t|}.$$

Consequently

$$h_\gamma(s,r+t) \le \gamma^{-2}(\gamma+|t|)^2 h_\gamma(s,r)$$

when $s,\ t,\ r\in\mathbb{R}$.

Proof. The lemma follows from the inequalities

$$1+|s-t| \ge 1+\frac{|s-t|}{1+|t|} \ge 1+\frac{|s|-|t|}{1+|t|} = \frac{1+|s|}{1+|t|}.$$

$\square$

The estimate of F_N that we need is contained in the next lemma.

Lemma 3.6. *There is a constant C, which does not depend on N and γ, such that*

$$|F_N(r_1,\dots,r_N)| \le C^N N^{2N+1}\gamma^{-(2N+1)}\mathrm{e}^{2\gamma}\prod_{1\le j\le N-1} h_\gamma(r_j,r_N). \tag{3.11}$$

Proof. It follows from (3.6) and (2.2) that

$$F_N(r_1,\dots,r_N) = (-1)^{N-1}\int_{-\infty}^{\infty}\Phi_N(r_1,\dots,r_N,t)\chi_N(t)\ \mathrm{d}t \tag{3.12}$$

where

$$\Phi_N(r_1,\dots,r_N,t) = (\varphi_{r_1} * \cdots * \varphi_{r_{N-1}})(t)\cos(tr_N).$$

As a function of t, $\Phi_N(r_1,\dots,r_N,t)$ is supported in $[0,\infty)$ and of polynomial growth at infinity.

Define

$$\widetilde{\Phi}_{N,\gamma}(r_1,\dots,r_N,t) = \mathrm{e}^{-\gamma t}\Phi_N(r_1,\dots,r_N,t), \quad \widetilde{\chi}_{N,\gamma}(t) = \mathrm{e}^{\gamma t}\chi_N(t).$$

Then

$$\begin{aligned} F_N(r_1,\dots,r_N) &= \int_{\mathbb{R}} \widetilde{\Phi}_{N,\gamma}(r_1,\dots,r_N,t)\widetilde{\chi}_{N,\gamma}(t)\ \mathrm{d}t \\ &= (2\pi)^{-1}\int_{\mathbb{R}} (\mathcal{F}\widetilde{\Phi}_{N,\gamma})(r_1,\dots,r_N,\tau)(\mathcal{F}\widetilde{\chi}_{N,\gamma})(-\tau)\ \mathrm{d}\tau, \end{aligned} \tag{3.13}$$

where the Fourier transform is taken in the variable t. We notice that

$$(\mathcal{F}\widetilde{\Phi}_{N,\gamma})(r_1,\dots,r_N,\tau) = \int \Phi_N(r_1,\dots r_N,t)\mathrm{e}^{-\sigma t}\,\mathrm{d}t = F(r_1,\dots,r_N;\sigma), \quad \sigma = \gamma + \mathrm{i}\tau.$$

Then an application of Lemma 3.3 gives the estimate

$$\begin{aligned} &|(\mathcal{F}\widetilde{\Phi}_{N,\gamma})(r_1,\dots,r_N,\tau)| \\ &\le \frac{1}{2}\prod_{1\le j\le N-1}|r_j^2-(r_N+\mathrm{i}\sigma)^2|^{-1} + \frac{1}{2}\prod_{1\le j\le N-1}|r_j^2-(r_N-\mathrm{i}\sigma)^2|^{-1} \\ &= \frac{1}{2}\prod_{1\le j\le N-1}|r_j-(r_N-\tau)-\mathrm{i}\gamma|^{-1}|r_j+(r_N-\tau)+\mathrm{i}\gamma|^{-1} \\ &+\frac{1}{2}\prod_{1\le j\le N-1}|r_j-(r_N+\tau)-\mathrm{i}\gamma|^{-1}|r_j+(r_N+\tau)+\mathrm{i}\gamma|^{-1} \\ &\le 2^{N-2}\prod_{1\le j\le N-1}(\gamma+|r_j-(r_N-\tau)|)^{-1}(\gamma+|r_j+(r_N-\tau)|)^{-1} \\ &+2^{N-2}\prod_{1\le j\le N-1}(\gamma+|r_j-(r_N+\tau)|)^{-1}(\gamma+|r_j+(r_N+\tau)|)^{-1} \\ &= 2^{N-2}\prod_{1\le j\le N-1}h_\gamma(r_j,r_N-\tau) + 2^{N-2}\prod_{1\le j\le N-1}h_\gamma(r_j,r_N+\tau). \end{aligned} \tag{3.14}$$

Next we see that

$$\begin{aligned} \mathcal{F}\widetilde{\chi}_{N,\gamma}(-\tau) &= \int \chi_{N,\gamma}(t)\mathrm{e}^{t(\gamma+\mathrm{i}\tau)}\ \mathrm{d}t \\ &= (\gamma+\mathrm{i}\tau)^{-(2N+2)}\int \chi_{N,\gamma}^{(2N+2)}(t)\mathrm{e}^{t(\gamma+\mathrm{i}\tau)}\ \mathrm{d}t. \end{aligned}$$

From this and Lemma 3.4 we deduce that there is a constant C, which is independent of N and γ, such that

$$|\mathcal{F}\widetilde{\chi}_{N,\gamma}(-\tau)| \le C^N N^{2N+2}\mathrm{e}^{2\gamma}(\gamma+|\tau|)^{-2N-2}.$$

Then (3.13), (3.14) and the above inequality, together with Lemma 3.5, give

$$
\begin{aligned}
|F_N(r_1,\dots,r_N)| &\le C^N N^{2N+2}\mathrm{e}^{2\gamma}\int_{-\infty}^{\infty}(\gamma+|\tau|)^{-2N-2}\Big(\prod_{1\le j\le N-1}h_\gamma(r_j,r_N-\tau)\Big)\,\mathrm{d}\tau\\
&\le C^N N^{2N+2}\gamma^{-2(N-1)}\mathrm{e}^{2\gamma}\big(\int_{-\infty}^{\infty}(\gamma+|\tau|)^{-4}\,\mathrm{d}\tau\big)\Big(\prod_{1\le j\le N-1}h_\gamma(r_j,r_N)\Big)\\
&\le C^N\gamma^{-(2N+1)}N^{2N+2}\mathrm{e}^{2\gamma}\prod_{1\le j\le N-1}h_\gamma(r_j,r_N)\\
&\le (2C)^N\gamma^{-(2N+1)}N^{2N+1}\mathrm{e}^{2\gamma}\prod_{1\le j\le N-1}h_\gamma(r_j,r_N).
\end{aligned}
$$

This finishes the proof. □

The following theorem gives the estimate we need for the Fourier transform of $E_{N,R}$.

Theorem 3.7. *There is a constant $C>0$, which depends on n only, such that*

$$
|(\mathcal{F}E_{N,R})(\xi_1,\dots,\xi_N)|\le C^N(N/(R\gamma))^{2N+1}e^{2R\gamma}\prod_{1\le j\le N-1}h_\gamma(|\xi_j|,|\xi_N|),\qquad \xi_1,\dots\xi_N\in\mathbb{R}^n
$$

for every $N\ge 2$, $R>0$ and $\gamma>0$.

Proof. Let $R>0$. The identity (3.9) and previous lemma show that there is a constant $C>0$, which depends on n only, such that

$$
|(\mathcal{F}E_{N,R})(\xi_1,\dots,\xi_N)|\le C^N(N/\gamma)^{2N+1}R^{2N-2}e^{2\gamma}\prod_{1\le j\le N-1}h_\gamma(R|\xi_j|,R|\xi_N|),
$$

when $\xi_1,\dots,\xi_N\in\mathbb{R}^n$, for every $N\ge 2$, $R>0$ and $\gamma>0$. This in turn shows that, with the same C, one has

$$
|(\mathcal{F}E_{N,R})(\xi_1,\dots,\xi_N)|\le C^N(N/\gamma)^{2N+1}e^{2\gamma}\prod_{1\le j\le N-1}h_{\gamma/R}(|\xi_j|,|\xi_N|).
$$

The theorem follows by replacing γ/R by γ. □

4. L^2-Sobolev estimates for B_N

We introduce an N-linear version of B_N, $N\ge 2$. Namely, for $\vec{v}=(v_1,\dots,v_N)$, $v_j\in C_0^\infty(\mathbb{R}^n)$, define

$$
\begin{aligned}
(\mathbf{B}_N\vec{v})(x) = &\int_{(\mathbb{R}^n)^N} E_N(y_1,\dots,y_N)\\
&v_1(x-\frac{y_N}{2}-Y_0)v_2(x-\frac{y_N}{2}-Y_1)\cdots v_N(x-\frac{y_N}{2}-Y_{N-1})\,\mathrm{d}\vec{y}.
\end{aligned}
\tag{4.1}
$$

Here the Y_k:s are defined as in Proposition 2.2, that is,

$$Y_0 = \frac{1}{2}\sum_1^{N-1} y_j, \qquad Y_k = Y_0 - \sum_{j=1}^{k} y_j, \quad k = 1, \dots, N-1.$$

Then $\mathbf{B}_N\vec{v}$ is a smooth compactly supported function in $\mathbb{R}^n$ and $B_N v = \mathbf{B}_N(v, \dots, v)$ for every $v \in C_0^\infty(\mathbb{R}^n)$. Therefore the result in Theorem 1.2 is contained in the next theorem. Here and in the rest of the section we use the notation

$$m = \frac{n-3}{2}.$$

Theorem 4.1. *Assume that* $0 < \varepsilon < 1$, $s_j \geq m$ *and* $a_j = \min(s_j - m, 1-\varepsilon)$, $j = 1, \dots, N$. *Set*

$$\sigma = \min(s_j - a_j) + \sum_{j=1}^{N} a_j.$$

Then there is a constant C *which is independent of the* s_j, *but may depend on* ε *and* n, *such that*

$$\|\mathbf{B}_N\vec{v}\|^2_{H_{(\sigma)}(B(0,R))} \leq C^N N^{2\min(s_j - a_j - m)}(R/N)^{N-1}\prod_1^N \|v_j\|^2_{(s_j)}, \tag{4.2}$$

for every $N \geq 2$, $R > 0$, $v_1, \dots, v_N \in C_0^\infty(B(0,R))$.

The present section is devoted to the proof of the above result. We start with some preparations.

Let $R > 0$ and recall that the distributions $E_{N,R} \in \mathcal{E}'((\mathbb{R}^n)^N)$ were defined in (3.6). When $\vec{v} = (v_1, \dots, v_N)$, $v_j \in C_0^\infty(\mathbb{R}^n)$, we consider

$$\begin{aligned}(\mathbf{B}_{N,R}\vec{v})(x) = &\int_{(\mathbb{R}^n)^N} E_{N,R}(y_1, \dots, y_N) \\ &v_1(x - \frac{y_N}{2} - Y_0)v_2(x - \frac{y_N}{2} - Y_1)\cdots v_N(x - \frac{y_N}{2} - Y_{N-1})\,\mathrm{d}\vec{y}.\end{aligned} \tag{4.3}$$

It is easy to see that $\mathbf{B}_{N,R}\vec{v}$ is a smooth compactly supported function in $\mathbb{R}^n$. The following lemma gives the connection between $\mathbf{B}_{N,R}\vec{v}$ and $\mathbf{B}_N\vec{v}$.

Lemma 4.2. *Assume* $v_1, \dots, v_N \in C_0^\infty(B(0,R))$. *Then* $(\mathbf{B}_{N,4R}\vec{v})(x) = (\mathbf{B}_N\vec{v})(x)$ *in a neighbourhood of* $\overline{B(0,R)}$ *and* $\mathbf{B}_{N,2(N-1)R}\vec{v} = \mathbf{B}_N\vec{v}$.

Proof. Choose $\varepsilon > 0$ such that the v_j are supported in $B(0, R-\varepsilon)$ and define

$$V_x(\vec{y}) = v_1(x - y_N/2 - Y_0)\cdots v_N(x - y_N/2 - Y_{N-1}). \tag{4.4}$$

Since $Y_0 + Y_{N-1} = 0$, it follows that

$$|2x - y_N| = |(x - y_N/2 - Y_0) + (x - y_N/2 - Y_{N-1})| \leq 2R - 2\varepsilon$$

when $\vec{y} \in \operatorname{supp}(V_x)$. When $|x| < R + \varepsilon/2$ we see that $|y_N| < 4R$ when $\vec{y}$ is in the support of V_x and, since $E_{N,4R} = E_N$ when $|y_N| < 4R$, it follows that $(\mathbf{B}_{N,4R}\vec{v})(x) = (\mathbf{B}_N\vec{v})(x)$ when $|x| < R + \varepsilon/2$. This proves the first assertion. When proving the second assertion we notice that

$$|y_j| = |(x - y_N/2 - Y_{j-1}) - (x - y_N/2 - Y_j)| < 2R$$

when $1 \le j \le N-1$ and $\vec{y} \in \operatorname{supp}(V_x)$, hence $\sum_1^{N-1} |y_j| < 2(N-1)R$. This shows that the support of V_x does not intersect the support of $E_{N,2(N-1)R} - E_N$, hence $\mathbf{B}_{N,2(N-1)R}\vec{v} = \mathbf{B}_N\vec{v}$. □

Let $\vec{s} = (s_1, \dots, s_N)$ be a sequence of nonnegative real numbers and let $\sigma \in \mathbb{R}$, $N \ge 2$, $R > 0$. Define

$$A(N, R, \vec{s}, \sigma) = \sup_{\xi_N} \int \cdots \int (1 + 4|\xi_N|^2)^\sigma |(\mathcal{F}E_{N,R})(\xi_1, \dots, \xi_N)|^2 M_{\vec{s}}(\xi_1, \dots, \xi_N)^2 \, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{N-1}, \tag{4.5}$$

where

$$M_{\vec{s}}(\xi_1, \dots, \xi_N) = \langle \xi_1 + \xi_N \rangle^{-s_1} \langle \xi_2 - \xi_1 \rangle^{-s_2} \cdots \langle \xi_N - \xi_{N-1} \rangle^{-s_N}.$$

Then $0 \le A(N, R, \vec{s}, \sigma) \le \infty$.

Lemma 4.3. *We have that*

$$\|\mathbf{B}_{N,R}\vec{v}\|_{(\sigma)}^2 \le (2\pi)^{n(1-N)} A_{N,R}(s_1, \dots, s_N, \sigma) \prod_1^N \|v_j\|_{(s_j)}^2 \tag{4.6}$$

for every $v_j \in C_0^\infty(\mathbb{R}^n)$, $1 \le j \le N$.

Proof. Let V_x be defined as in (4.4). In order to compute the Fourier transform of V_x we introduce the linear map L in $(\mathbb{R}^n)^N$ through $L\vec{z} = \vec{y}$, where

$$y_j = z_j - z_{j+1}, \ 1 \le j \le N-1, \ y_N = z_1 + z_N.$$

It is easily seen that $\det(L) = 2^n$ and that $y_N/2 + Y_{j-1} = z_j$ when $1 \le j \le N$. Therefore we may write

$$V_x(\vec{y}) = (v_1 \otimes \cdots \otimes v_N)(-L^{-1}(y_1, \dots y_{N-1}, y_N - 2x)).$$

Hence

$$\mathcal{F}V_x(-\xi_1, \dots, -\xi_N) = 2^n \mathrm{e}^{2\mathrm{i}\langle x, \xi_N \rangle} (\hat{v}_1 \otimes \cdots \otimes \hat{v}_N)(L'\vec{\xi}).$$

Here L' denotes the transpose of L. It is easy to see that $L'\vec{\xi} = \vec{\eta}$, where

$$\eta_1 = \xi_1 + \xi_N, \ \eta_j = \xi_j - \xi_{j-1}, \quad 2 \le j \le N.$$

It follows that

$$(\mathcal{F}V_x)(-\xi_1, \dots, -\xi_N) = 2^n \mathrm{e}^{2\mathrm{i}\langle x, \xi_N \rangle} \widehat{v}_1(\xi_1 + \xi_N) \widehat{v}_2(\xi_2 - \xi_1) \cdots \widehat{v}_N(\xi_N - \xi_{N-1}).$$

Write $w_j = \mathcal{F}\langle D\rangle^{s_j} v_j$ and

$$W(\vec{\xi}) = w_1(\xi_1 + \xi_N) w_2(\xi_2 - \xi_1) \cdots w_N(\xi_N - \xi_{N-1}).$$

It follows from (4.3) and the computations above that

$$\begin{aligned}(\mathbf{B}_{N,R}\vec{v})(x) &= (2\pi)^{-nN} \int (\mathcal{F}E_{N,R})(\vec{\xi})(\mathcal{F}V_x)(-\vec{\xi})\,\mathrm{d}\vec{\xi} \\ &= (2\pi)^{-nN} 2^n \int \mathrm{e}^{2\mathrm{i}\langle x, \xi_N\rangle} \beta(\xi_N)\,\mathrm{d}\xi_N \\ &= (2\pi)^{-nN} \int \mathrm{e}^{\mathrm{i}\langle x, \xi_N\rangle} \beta(\xi_N/2)\,\mathrm{d}\xi_N,\end{aligned}$$

where

$$\begin{aligned}\beta(\xi_N) &= \int \cdots \int (\mathcal{F}E_{N,R})(\vec{\xi}) \widehat{v}_1(\xi_1 + \xi_N) \widehat{v}_2(\xi_2 - \xi_1) \cdots \widehat{v}_N(\xi_N - \xi_{N-1})\,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{N-1} \\ &= \int \cdots \int (\mathcal{F}E_{N,R})(\vec{\xi}) M_{\vec{s}}(\vec{\xi}) W(\vec{\xi})\,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{N-1}.\end{aligned}$$

This shows that

$$\begin{aligned}\|\mathbf{B}_{N,R}\vec{v}\|^2_{(\sigma)} &= (2\pi)^{-2n(N-1/2)} \int \langle \xi_N\rangle^{2\sigma} |\beta(\xi_N/2)|^2\,\mathrm{d}\xi_N \\ &= 2^n (2\pi)^{-2n(N-1/2)} \int (1 + |4\xi_N|^2)^\sigma |\beta(\xi_N)|^2\ \mathrm{d}\xi_N \\ &\le 2^n (2\pi)^{-2n(N-1/2)} \int \Big\{ (1 + 4|\xi_n|^2)^\sigma \Big(\int \cdots \int |(\mathcal{F}E_{N,R})(\vec{\xi})|^2 M_{\vec{s}}(\xi)^2\,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{N-1} \Big) \\ &\qquad \Big(\int \cdots \int |W(\vec{\xi})|^2\,\mathrm{d}\xi_1 \cdots d\xi_{N-1} \Big) \Big\}\,\mathrm{d}\xi_N. \\ &\le 2^n (2\pi)^{-2n(N-1/2)} A(N, R, \vec{s}, \sigma) \int |W(\vec{\xi})|^2\,\mathrm{d}\vec{\xi}.\end{aligned} \tag{4.7}$$

The proof is then completed by the observation that

$$W(\vec{\xi}) = (w_1 \otimes w_2 \otimes \cdots \otimes w_N)(L'\vec{\xi}).$$

It follows that

$$\int |W(\vec{\xi})|^2\,\mathrm{d}\vec{\xi} = 2^{-n} \int |(w_1 \otimes \cdots \otimes w_N)(\vec{\xi})|^2\,\mathrm{d}\vec{\xi}$$

$$= 2^{-n} \prod_1^N \|w_j\|^2 = 2^{-n}(2\pi)^{nN} \prod_1^N \|\langle D\rangle^{s_j} v_j\|^2 = 2^{-n}(2\pi)^{nN} \prod_1^N \|v_j\|^2_{(s_j)}.$$

The lemma follows if this is inserted into (4.7). □

We shall arrive at estimates for $B_{N,R}\vec{v}$ by combining the inequality (4.6) with estimates for the expression $A(N, R, \vec{s}, \sigma)$ in (4.5). The following lemma will be needed.

Lemma 4.4. *Assume* $0 < \varepsilon < 1$*. Then there is a constant* $C = C_{n,\varepsilon}$ *such that*

$$\int h_\gamma^2(|\xi|, \rho) \langle \xi - \eta\rangle^{-2s}\,\mathrm{d}\xi \le C\gamma^{-1} \langle \rho\rangle^{2m-2s}, \tag{4.8}$$

when $\eta \in \mathbb{R}^n$*,* $\rho \ge 0$*,* $\gamma > 0$*,* $m \le s \le m + 1 - \varepsilon$*.*

Proof. Assume $r > 0$ and $\eta \in \mathbb{R}^n \setminus 0$. Set

$$f_s(r,\eta) = \int\limits_{\mathbb{S}^{n-1}} \langle r\theta - \eta\rangle^{-2s} \, \mathrm{d}\theta.$$

If $u = \langle \theta, \eta\rangle / |\eta|$ then a simple computation shows that

$$\langle r\theta - \eta\rangle^2 \geq 1 + r^2(1 - |u|).$$

If f is a continuous function, and c_{n-2} is the area of the $n-2$-dimensional unit sphere, then

$$\int\limits_{\mathbb{S}^{n-1}} f(\langle \theta, \eta\rangle / |\eta|) \, \mathrm{d}\eta = c_{n-2} \int_{-1}^{1} f(t)(1-t^2)^m \, \mathrm{d}t.$$

This shows that

$$f_s(r,\eta) \leq c_{n-2} \int\limits_{-1}^{1} (1 + r^2(1 - |t|))^{-s}(1-t^2)^m \, \mathrm{d}t$$

$$\leq 2^{m+1} c_{n-2} \int\limits_{0}^{1} (1 + r^2(1-t))^{-s}(1-t)^m \, \mathrm{d}t \leq 2^{m+1} c_{n-2} \int\limits_{0}^{1} (1 + r^2 t)^{-s} t^m \, \mathrm{d}t$$

$$\leq 2^{m+1} c_{n-2} \langle r\rangle^{-2s} \int\limits_{0}^{1} t^{m-s} \, \mathrm{d}t.$$

This gives the estimate

$$f_s(r,\eta) \leq C_1 \langle r\rangle^{-2s}, \tag{4.9}$$

where $C_1 = 2^{m+1} c_{n-2} / \varepsilon$, for $\eta \in \mathbb{R}^n \setminus 0$. This inequality clearly holds for $\eta = 0$ as well.

Using (4.9) and introducing polar coordinates in the integration one gets

$$\int h_\gamma^2(|\xi|, \rho) \langle \xi - \eta\rangle^{-2s} \, \mathrm{d}\xi \leq C_1 \int\limits_{0}^{\infty} h_\gamma^2(r,\rho) r^{2m+2} \langle r\rangle^{-2s} \, \mathrm{d}r. \tag{4.10}$$

Assume first that $\rho \geq 1$. Then

$$\begin{aligned}
&\int\limits_{0}^{\infty} h_\gamma^2(r,\rho) r^{2m+2} \langle r\rangle^{-2s} \, \mathrm{d}r \\
&= \int\limits_{0}^{\infty} \frac{1}{(\gamma + |r-\rho|)^2} \frac{1}{(\gamma + r + \rho)^2} \frac{r^{2m+2}}{(r^2+1)^s} \, \mathrm{d}r \\
&\leq \frac{1}{\rho^{2(s-m)}} \int\limits_{0}^{\infty} \frac{1}{(\gamma + |r-\rho|)^2} \frac{r^{2m+2}}{(r^2+1)^{m+1}} \, \mathrm{d}r \\
&\leq \frac{2^{2(s-m)}}{(\rho+1)^{2(s-m)}} \int\limits_{\mathbb{R}} \frac{1}{(\gamma + |r|)^2} \, \mathrm{d}r \leq 2^3 \gamma^{-1} \langle \rho\rangle^{-2(s-m)}.
\end{aligned} \tag{4.11}$$

Assume next that $\rho < 1$. Then

$$\begin{aligned}
&\int_0^\infty h_\gamma^2(r,\rho) r^{2m+2} \langle r\rangle^{-2s}\,\mathrm{d}r \\
&\le \int_0^\infty \frac{1}{(\gamma+|r-\rho|)^2}\,\frac{r^{2m}}{(r^2+1)^s}\,\mathrm{d}r \le \int_{\mathbb{R}} \frac{1}{(\gamma+|r|)^2}\,\mathrm{d}r \\
&\le 2^{2(s-m)+1}\gamma^{-1}(\rho^2+1)^{-(s-m)} \le 2^3\gamma^{-1}\langle\rho\rangle^{-2(s-m)}.
\end{aligned} \tag{4.12}$$

Combining (4.10), (4.11) and (4.12) we see that the lemma holds with $C = 2^3 C_1$. □

Now we are going to estimate $A(N,R,\vec{s},\gamma)$. Recall that Theorem 3.7 gives that

$$|(\mathcal{F}E_{N,R})(\xi_1,\ldots,\xi_N)| \le \; C^N (N/(\gamma R))^{2N+1} e^{2R\gamma} \prod_{1\le j\le N-1} h_\gamma(|\xi_j|,|\xi_N|), \tag{4.13}$$

where $\gamma > 0$, $N \ge 2$, $R > 0$ and the constant C is independent of these parameters. We notice that

$$\begin{aligned}
M_{\vec{s}}(\xi_1,\ldots,\xi_N) &\le 2^{s_2}\langle\xi_1+\xi_N\rangle^{-s_1} M_{s_2,\ldots,s_N}(\xi_2,\ldots,\xi_N) \\
&\quad + 2^{s_1}\langle\xi_2-\xi_1\rangle^{-s_2} M_{s_1,s_3,\ldots,s_N}(\xi_2,\ldots,\xi_N).
\end{aligned} \tag{4.14}$$

In fact, since

$$|\xi_2+\xi_N| \le |\xi_2-\xi_1| + |\xi_1+\xi_N|,$$

either $|\xi_2-\xi_1| \ge |\xi_2+\xi_N|/2$ or $|\xi_1+\xi_N| \ge |\xi_2+\xi_N|/2$. In the first case

$$M_{\vec{s}}(\xi_1,\ldots,\xi_N) \le 2^{s_2}\langle\xi_1+\xi_N\rangle^{-s_1} M_{s_2,\ldots,s_N}(\xi_2,\ldots,\xi_N)$$

and in the second case

$$M_{\vec{s}}(\xi_1,\ldots,\xi_N) \le 2^{s_1}\langle\xi_1-\xi_2\rangle^{-s_2} M_{s_1,s_3,\ldots,s_N}(\xi_2,\ldots,\xi_N).$$

When $N \ge 2$ we define

$$T_{N,\gamma}(\xi,\vec{s}) = \int\cdots\int \Big(\prod_{1\le j\le N-1} h_\gamma(|\xi_j|,|\xi|)\Big)^2 M_{\vec{s}}^2(\xi_1,\ldots,\xi_{N-1},\xi)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\ldots\,\mathrm{d}\xi_{N-1}.$$

Let $0 < \varepsilon < 1$ and assume that $m \le s_j \le m+1-\varepsilon$ when $0 \le j \le N$. It follows from (4.13) and (4.5) that

$$A(N,R,\vec{s},\sigma) \le C^N (N/(\gamma R))^{4N+2} e^{4R\gamma} \sup_\xi (\langle 2\xi\rangle^{2\sigma} T_{N,\gamma}(\xi;\vec{s})). \tag{4.15}$$

Here, and in what follows, C denotes constants that are independent of N, R, $\vec{s}$, σ, γ (but may depend on ε and dimension n).

Assume $N \geq 3$. From (4.14) follows that

$$\begin{aligned} T_{N,\gamma}(\xi;\vec{s}) \leq 2^n\Big(\int h_\gamma^2(|\xi_1|,|\xi|)\langle\xi_1+\xi\rangle^{-2s_1}\,\mathrm{d}\xi_1\big) \\ \int\cdots\int\Big(\prod_{2\leq j\leq N-1} h_\gamma^2(|\xi_j|,|\xi|)\Big)M^2_{s_2,\dots,s_N}(\xi_2,\dots,\xi_{N-1},\xi)\,\mathrm{d}\xi_2\cdots\,\mathrm{d}\xi_{N-1} \\ +2^n\Big(\int h_\gamma^2(|\xi_1|,|\xi|)\langle\xi_1-\xi_2\rangle^{-2s_2}\,\mathrm{d}\xi_1\big) \\ \int\cdots\int\Big(\prod_{2\leq j\leq N-1} h_\gamma^2(|\xi_j|,|\xi|)\Big)M^2_{s_1,s_3,\dots,s_N}(\xi_2,\dots,\xi_{N-1},\xi)\,\mathrm{d}\xi_2\cdots\,\mathrm{d}\xi_{N-1}. \end{aligned}$$

From Lemma 4.4 we get the estimate

$$\begin{aligned} T_{N,\gamma}(\xi;\vec{s}) &\leq (C/2)\gamma^{-1}\langle\xi\rangle^{2m-2s_1}T_{N-1,\gamma}(\xi;s_2,\dots,s_N) \\ &+(C/2)\gamma^{-1}\langle\xi\rangle^{2m-2s_2}T_{N-1,\gamma}(\xi;s_1,s_3,\dots,s_N). \end{aligned} \tag{4.16}$$

Another application of Lemma 4.4 gives

$$\begin{aligned} T_{2,\gamma}(\xi;s_1,s_2) = \int h_\gamma^2(|\xi_1|,|\xi|)\langle\xi_1+\xi\rangle^{-2s_1}\langle\xi_1-\xi\rangle^{-2s_2}\,\mathrm{d}\xi_1 \\ \leq \langle\xi\rangle^{-2s_1}\int h_\gamma^2(|\xi_1|,|\xi|)\langle\xi_1-\xi\rangle^{-2s_2}\,\mathrm{d}\xi_1 + \langle\xi\rangle^{-2s_2}\int h_\gamma^2(|\xi_1|,|\xi|)\langle\xi_1+\xi\rangle^{-2s_1}\,\mathrm{d}\xi_1 \\ \leq C\gamma^{-1}\langle\xi\rangle^{2m-2s_1-2s_2} \end{aligned}$$

where we may assume that C is the same constant as in (4.16). From this we deduce that the inequality

$$T_{N,\gamma}(\xi;\vec{s}) \leq C^{N-1}\gamma^{-(N-1)}\langle\xi\rangle^{2((N-1)m-s_1-\dots-s_N)} \tag{4.17}$$

holds when $N = 2$. Applying (4.16) together with an induction argument we obtain that (4.17) holds for every $N \geq 2$. Since $m \leq s_j < m+1$ we get (with another C)

$$T_{N,\gamma}(\xi;\vec{s}) \leq C^{N-1}\gamma^{-(N-1)}\langle 2\xi\rangle^{2((N-1)m-s_1-\dots-s_N)}. \tag{4.18}$$

Assume now that $s_j \geq m$, $j = 1,\dots,N$, but not necessarily $s_j < m+1$, and let $0 < \varepsilon < 1$. Set $a_j = \min(s_j - m, 1-\varepsilon)$, $j = 1,\dots,N$. We notice that

$$|\xi_1+\xi_N| + |\xi_2-\xi_1| + \dots + |\xi_N - \xi_{N-1}| \geq 2|\xi_N|,$$

and therefore

$$\max(|\xi_1+\xi_N|, |\xi_2-\xi_1|, \cdots |\xi_N-\xi_{N-1}|) \geq 2|\xi_N|/N.$$

It follows that

$$\langle\xi_1+\xi_N\rangle^{-1}\langle\xi_2-\xi_1\rangle^{-1}\cdots\langle\xi_N-\xi_{N-1}\rangle^{-1} \leq (1+4|\xi_N|^2/N^2)^{-1/2} \leq N\langle 2\xi_N\rangle^{-1}.$$

Then we may write

$$M_{\vec{s}}(\vec{\xi}) \leq N^{\min(s_j-(a_j+m))}\langle 2\xi_N\rangle^{-\min(s_j-(a_j+m))}M_{(m+a_1,\dots,m+a_N)}(\vec{\xi}).$$

This implies that

$$T_{N,\gamma}(\xi;\vec{s}) \leq N^{2\min(s_j-(a_j+m))}\langle 2\xi\rangle^{-2\min(s_j-(a_j+m))}T_{N,\gamma}(\xi; m+a_1,\ldots,m+a_N).$$

Then (4.18) gives

$$T_{N,\gamma}(\xi,\vec{s}) \leq C^{N-1}N^{2\min(s_j-(a_j+m))}\gamma^{-(N-1)}\langle 2\xi\rangle^{-2\min(s_j-a_j)-2\sum_1^N a_j}. \tag{4.19}$$

Combining (4.15) with (4.19) we get the following lemma.

Lemma 4.5. *Assume that $0<\varepsilon<1$, $s_j \geq m$ and $a_j = \min(s_j-m, 1-\varepsilon)$, $j=1,\ldots,N$. Set*

$$\sigma = \min(s_j-a_j) + \sum_{j=1}^N a_j$$

Then there is a constant C which is independent of the s_j, but may depend on ε and n, such that

$$A(N,R,\vec{s},\sigma) \leq C^N N^{2\min(s_j-a_j-m)}(N/(\gamma R))^{4N+2}\gamma^{-(N-1)}e^{4R\gamma} \tag{4.20}$$

for every $\gamma>0$, $R>0$ and $N\geq 2$.

Next we recall (4.6) which, together with the previous lemma, gives the next proposition.

Proposition 4.6. *Assume that $0<\varepsilon<1$, $s_j \geq m$ and $a_j = \min(s_j-m, 1-\varepsilon)$, $j=1,\ldots,N$. Set*

$$\sigma = \min(s_j-a_j) + \sum_{j=1}^N a_j.$$

Then there is a constant C which is independent of the s_j, but may depend on ε and n, such that

$$\|\mathbf{B}_{N,R}\vec{v}\|^2_{(\sigma)} \leq C^N N^{2\min(s_j-a_j-m)}(N/(\gamma R))^{4N+2}\gamma^{-(N-1)}e^{4R\gamma}\prod_1^N \|v_j\|^2_{(s_j)}, \tag{4.21}$$

for every $N\geq 2$, $v_1,\ldots,v_N \in C_0^\infty(\mathbb{R}^n)$, $R>0$ and $\gamma>0$.

Theorem 4.1 follows from the previous proposition and Lemma 4.2, by replacing R by $4R$ and taking $\gamma = N/(4R)$. When replacing R by $2(N-1)R$ and taking $\gamma = 1/R$, we obtain the following corollary, where we use Lemma 4.2.

Corollary 4.7. *Assume that $0<\varepsilon<1$, $s_j \geq m$ and $a_j = \min(s_j-m, 1-\varepsilon)$, $j=1,\ldots,N$. Set*

$$\sigma = \min(s_j-a_j) + \sum_{j=1}^N a_j.$$

Then there is a constant C, which depends on n, ε and the s_j only, such that

$$\|\mathbf{B}_N\vec{v}\|^2_{(\sigma)} \leq C^N R^{N-1} \prod_1^N \|v_j\|^2_{(s_j)}, \tag{4.22}$$

for every $N \geq 2$, $R > 0$ and $v_1, \dots, v_N \in C_0^\infty(B(0,R))$.

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